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UNSTEADY BOUNDARY LAYER IN IMPULSIVE STAGNATION FLOW

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NOMENCLATURE

C,	constant;
f,	stream function;
i,	variable defined to be zero for plane flow and one for axisymmetric flow;
K,	constant;
L,	characteristic length of the body;
p,	Laplace transform variable;
Pr,	Prandtl number;
t,	time;
T,	temperature;
T_∞,	temperature at infinity;
T_w,	temperature on the surface;
u,	velocity in x direction;
u_p,	potential inviscid velocity on the surface;
v,	velocity in y direction;
x,	coordinate along the body;
y,	coordinate normal to the body.

Greek symbols

α,	thermal diffusivity;
η,	variable defined by (7);
θ,	nondimensional temperature;
ϑ,	nondimensional temperature;
$\bar{\vartheta}$,	Laplace transform of ϑ;
ν,	kinematic viscosity;
ξ,	variable defined by (15);
τ,	nondimensional time;
ψ,	stream function.

INTRODUCTION

THE PRESENT note is concerned with unsteady thermal boundary layer in the impulsive stagnation flow. Watkins [1] recently considered the more general problem of impulsive Falkner–Skan flows and obtained numerical solutions for moderate values of Prandtl number for the situation in which thermal boundary layer is produced by sudden imposition of a constant temperature difference between the body and the fluid as the impulsive motion is started. In the present note, analytical solutions for impulsive stagnation flow are obtained for two situations of thermal condition on the surface; one is the same situation as that treated by Watkins, that is, step change in wall temperature, and the other is the case of step change in wall heat flux. For the latter case, no solutions have been obtained up to now as far as the present writer knows. Present analysis is restricted only to low Prandtl number fluids such as liquid metals, which will be useful because of its possible application in the nuclear field.

GOVERNING EQUATIONS

Consider the flow field around a two-dimensional or rotationally symmetrical body which is set in a steady motion impulsively. In the neighbourhood of the front stagnation point, the inviscid potential flow described by

$$u_p = Kx, \tag{1}$$

where K is constant, is established instantaneously with the impulsive motion.

In the boundary layer, introducing the following variables

$$\left.\begin{aligned} \eta &= y(2^iK/\nu)^{1/2}, \quad \tau = 2^iKt, \\ \psi &= (\nu K/2^i)^{1/2}(x/L)^i x f(\tau,\eta), \end{aligned}\right\} \tag{2}$$

where ψ is the stream function defined by

$$\left.\begin{aligned} u &= (L/x)^i(\partial\psi/\partial y), \\ v &= -(L/x)^i(\partial\psi/\partial x), \end{aligned}\right\} \tag{3}$$

L being the characteristic length of the body, the governing equation for the velocity can be written as

$$\frac{\partial f'}{\partial \tau} + f''' + ff'' + 2^{-i}(1-f'^2) = 0 \tag{4}$$

with boundary conditions

$$\left.\begin{aligned} &\tau \leqslant 0 \quad f = f' = 0, \\ &\tau > 0 \quad \begin{cases} f = f' = 0 & \text{for } \eta = 0, \\ f' \to 1 & \text{as } \eta \to \infty, \end{cases} \end{aligned}\right\} \tag{5}$$

where prime denotes differentiation with respect to η. We assume that the body and the fluid are initially ($t \leqslant 0$) at the same temperature and, as the impulsive motion is started, a constant temperature difference between the body and the fluid or a constant heat flux ($= kC$) through the wall is suddenly imposed. The governing equations and the boundary conditions for the temperature field then can be written as

$$Pr\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\eta^2} + Prf\frac{\partial\theta}{\partial\eta}, \tag{6}$$

and $\tau \leqslant 0$ $\quad \theta = 0$ (7a)

$\tau > 0$ (a) for step change in wall temperature

$$\left.\begin{aligned} &\theta = 1 \quad \text{at } \eta = 0 \\ &\theta \to 0 \quad \text{as } \eta \to \infty \end{aligned}\right\}. \tag{7b}$$

(b) for step change in wall flux

$$\left.\begin{aligned} &\partial\theta/\partial\eta = -1 \quad \text{at } \eta = 0 \\ &\theta \to 0 \quad\quad\quad \text{as } \eta \to \infty \end{aligned}\right\}. \tag{7c}$$

respectively, where

$$\theta = \begin{cases} (T-T_\infty)/(T_w-T_\infty) & \text{(8a)} \\ \text{for step change in wall temperature} & \\ (T-T_\infty)/C(\nu/2^iK)^{1/2} & \text{(8b)} \\ \text{for step change in wall flux.} & \end{cases}$$

ASYMPTOTIC SOLUTIONS FOR $Pr \to 0$

We shall now proceed to obtain the asymptotic solution of (6) for $Pr \to 0$. When Pr is small, the ratio of the thickness of thermal boundary layer to that of momentum layer is very large and therefore the velocity field in the thermal boundary layer may be approximated by its asymptotic expansion for large η:

$$f \sim \eta + \beta(\tau), \tag{9}$$

where β is a function of τ. Substituting (9) into (6), we have

$$\frac{\partial\vartheta}{\partial\tau} = \frac{\partial^2\vartheta}{\partial\xi^2} + [\xi + \beta(\tau)Pr^{1/2}]\frac{\partial\vartheta}{\partial\xi}. \tag{10}$$

where

$$\xi = Pr^{1/2}\eta, \quad \vartheta(\tau, \xi) = \theta(\tau, \eta). \tag{11}$$

Since we consider the limit of $Pr \to 0$, the second term in the parentheses may be neglected and the equation (10) becomes

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \xi^2} + \xi \frac{\partial \vartheta}{\partial \xi}. \tag{10a}$$

Defining the Laplace transform of $\vartheta(\tau, \xi)$ in the usual manner; i.e.

$$\bar{\vartheta} = \int_0^\infty e^{-p\tau}\vartheta(\tau, \xi)\,d\tau, \tag{12}$$

(10a) becomes

$$\frac{\partial^2 \bar{\vartheta}}{\partial \xi^2} + \xi \frac{\partial \bar{\vartheta}}{\partial \xi} - p\bar{\vartheta} = 0, \tag{13}$$

with boundary conditions

(a) for step change in wall temperature

$$\left.\begin{array}{ll} \bar{\vartheta} = 1/p & \text{at } \xi = 0 \\ \bar{\vartheta} \to 0 & \text{as } \xi \to \infty, \end{array}\right\} \tag{14a}$$

(b) for step change in wall flux

$$\left.\begin{array}{ll} \partial\bar{\vartheta}/\partial\xi = -1/Pr^{1/2}p & \text{at } \xi = 0 \\ \bar{\vartheta} \to 0 & \text{as } \xi \to \infty. \end{array}\right\} \tag{14b}$$

The solution of (13) satisfying the boundary condition at infinity is [2]

$$\bar{\vartheta} = A\left[\frac{{}_1F_1\left(-\frac{p}{2}, \frac{1}{2}, -\frac{\xi^2}{2}\right)}{2^{p/2}\Gamma\left(\frac{p+2}{2}\right)} - \frac{\xi\,{}_1F_1\left(-\frac{p-1}{2}, \frac{3}{2}, -\frac{\xi^2}{2}\right)}{2^{(p-1)/2}\Gamma\left(\frac{p+1}{2}\right)}\right] \tag{15}$$

where A is an integration constant and ${}_1F_1$ is the confluent hypergeometric function. The boundary conditions on the surface determine A as

$$A = 2^{p/2}\Gamma\left(\frac{p+2}{2}\right)\Big/ p \tag{16a}$$

for step change in wall temperature and as

$$A = 2^{(p-1)/2}\Gamma\left(\frac{p+1}{2}\right)\Big/ Pr^{1/2}p \tag{16b}$$

for step change in wall flux.

For the case of step change in wall temperature, the temperature gradient at the surface can be obtained by taking inverse of $(\partial\bar{\vartheta}/\partial\xi)_{\xi=0}$. The result is [3]

$$-(\partial\theta/\partial\eta)_{\eta=0} = \left(\frac{2Pr}{\pi}\right)^{1/2}\frac{1}{(1-e^{-2\tau})^{1/2}}. \tag{17}$$

For $\tau \to 0$, (17) becomes

$$-(\partial\theta/\partial\eta)_{\eta=0} \sim (Pr/\pi\tau)^{1/2}. \tag{18}$$

This asymptotic behaviour is in agreement with the temperature gradient obtained from the Rayleigh's solution

$$\theta = \operatorname{erfc}[\eta/(4\tau/Pr)^{1/2}], \tag{19}$$

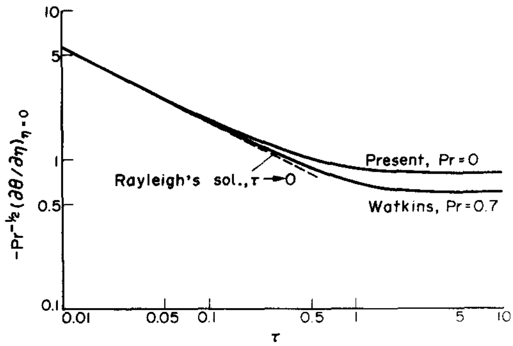

FIG. 1. Heat-transfer variation for step change in wall temperature.

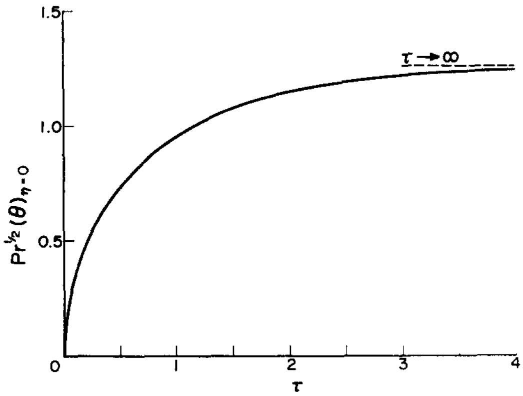

FIG. 2. Wall temperature variation for step change in wall flux.

which is valid for small τ. For $\tau \to \infty$, on the other hand, (17) becomes

$$-(\partial\theta/\partial\eta)_{\eta=0} \sim (2Pr/\pi)^{1/2}, \tag{20}$$

which is in agreement with the previously obtained steady state result for $Pr \to 0$. Figure 1 shows the heat-transfer result plotted as a function of time. For comparison, the result by Watkins for $Pr = 0.7$ and that from Rayleigh's solution are also shown in the figure.

For the case of step change in wall flux, the temperature at the surface can be obtained by taking inverse of $(\bar{\vartheta})_{\xi=0}$ as follows [3].

$$(\theta)_{\eta=0} = (2/Pr\pi)^{1/2}[\pi/2 - \sin^{-1}(e^{-\tau})]. \tag{21}$$

This relation is shown graphically in Fig. 2.

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